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Probabilistic finite element analysis in fluid mechanics

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Abstract *Fluid flow in a circular pipe and a slider bearing was computationally simulated by finite element methods and probabilistically evaluated in view of the several uncertainties in the performance parameters. Cumulative distribution functions and sensitivity factors were computed for the flow rate and load bearing capacity of the slider bearing due to the several random variables. These results can be used to quickly identify the most critical design variables in order to optimize the design and make it cost effective. The analysis leads to the selection of the appropriate measurements to be used in fluid flow and to the identification of both the most critical measurements and parameters.*

Introduction

Conventional engineering design methods are deterministic. Machines and their components are considered as ideal systems and parameter optimizations provide single point estimates of the system behavior or response. Probabilistic engineering design uses probability distributions of design parameters instead of mean or nominal values only. This will enable a designer for a specific reliability and hence maximize safety, quality and economy. A probabilistic design system was developed by Fox (1994) at Pratt and Whitney for the purpose of integrating deterministic design methods with probabilistic design techniques. Here, two different approaches were used for estimating uncertainty. A Monte Carlo approach was used on design codes that were judged to run relatively quickly. For more computationally intensive design codes, a second order response surface model in conjunction with Box-Behnken design experiments was used and then a Monte Carlo simulation was executed. Several researchers at NASA Glenn Research Center have applied the probabilistic design approaches to turbine engines and related systems. Chamis (1986a) developed a Probabilistic Structural Analysis Method (PSAM) using different distributions such as the Weibull, normal, log-normal etc. to describe the uncertainties in the structural and load parameters or primitive variables. Nagpal *et al.* (1987) presented a probabilistic study of turbopump blades of the Space Shuttle Main Engine (SSME). They found that random variations or uncertainties in geometry have statistically significant influence on the response variable and random variations in material properties have

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statistically insignificant effects. Chamis (1986b) summarized the usefulness and importance of the probabilistic approach, especially for turbopumps. Gorla *et al.* (2003) computationally simulated and probabilistically evaluated a combustor liner in view of several uncertainties in the aerodynamic, structural, material and thermal variables that govern the combustor liner.

To cost effectively accomplish the design task, we need to formally quantify the effect of uncertainties (variables) in the design. Probabilistic design is one effective method to formally quantify the effect of uncertainties. In the present paper, a probabilistic analysis is presented for the influence of measurement accuracy and *a priori* fixed parameter variations on the random variables for fluid flow in a circular pipe and a slider bearing. Small perturbation approach is used for the finite element methods to compute the sensitivity of the response to small fluctuations of the random variables present. The result is a parametric representation of the response in terms of a set of random variables with known statistical properties, which can be used to estimate the characteristics of the selected response variables such as flow rate and load carrying capacity of the bearing.

Analysis

Let us consider the flow of a fluid with viscosity μ through a circular pipe. A negative pressure gradient $\partial p/\partial z$ drives the flow in the positive z direction with axial velocity $w(x, y)$. The governing differential equation for the velocity profile is given by

$$\frac{\partial}{\partial x}\left(\mu\frac{\partial w}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right)-\frac{\partial p}{\partial z}=0 \qquad (1)$$

The no slip boundary condition is given by

$$w(x,y)=0 \text{ at } r=(x^2+y^2)^{1/2}=R \qquad (2)$$

The other boundary conditions are

$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial y}=0 \text{ at } x=0 \text{ and } y=0 \qquad (3)$$

In equation (2), R is the radius of the pipe.

The volume flow rate through the pipe is given by

$$Q=\iint_A w\,\mathrm{d}A=-\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) \qquad (4)$$

The second example to be considered is the flow of a viscous fluid in a slider bearing. The differential equation for the pressure $p(x, y)$ distribution in the bearing is given by the Reynolds equation:

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right)=6U\mu\frac{\partial h}{\partial x} \tag{5}$$

where $h = h(x)$ = oil film thickness and U = slider velocity

The boundary conditions are given by:

$$p = p_0 = \text{atmospheric pressure along all the four sides} \tag{6}$$

The force generated by the fluid film or the bearing load is given by

$$F=\iint_A p\cdot \mathrm{d}A \tag{7}$$

Finite element solution

Let us consider a two-dimensional partial differential equation of the form

$$\frac{\partial}{\partial x}\left[K_x(x,y)\frac{\partial T}{\partial x}\right]+\frac{\partial}{\partial y}\left[K_y(x,y)\frac{\partial T}{\partial y}\right]+P(x,y)T+Q(x,y)=0. \tag{8}$$

where T is the field variable to be determined.

The above equation is valid over an area A. We assume that on a portion of the boundary L_1, $T = T_0\,(x, y)$.

On the remainder of the boundary, labeled L_2, the general derivative boundary condition is specified in the form

$$K_x(x,y)\frac{\partial T}{\partial X}n_x+K_y(x,y)\frac{\partial T}{\partial y}n_y+\alpha(x,y)T+\beta(x,y)=0. \tag{9}$$

Here, n_x and n_y are direction cosines of the outward normal to L_2. The form of the functional may be written as

$$I(T)=\iint_A\left[\frac{1}{2}K_x\left(\frac{\partial T}{\partial x}\right)^2+\frac{1}{2}K_y\left(\frac{\partial T}{\partial y}\right)^2-\frac{1}{2}PT^2-QT\right]\mathrm{d}A + \int_{L_2}\left(\frac{\alpha T^2}{2}+\beta T\right)\mathrm{d}L. \tag{10}$$

For a simplex two-dimensional element, we have extremized the above functional with respect to the unknowns nodal values of the field variable. The resultant element matrices are then obtained from the following relation:

$$\begin{Bmatrix} \dfrac{\partial I}{\partial T_i} \\ \dfrac{\partial I}{\partial T_j} \\ \dfrac{\partial I}{\partial T_k} \end{Bmatrix}^{(e)} = [B]^{(e)}[T]^{(e)} - [C]^{(e)} \tag{11}$$

The element matrix $[B]^{(e)}$ and the element column $[C]^{(e)}$ may be written as

$$[B]^{(e)} = \begin{bmatrix} B_{ii}B_{ij}B_{ik} \\ B_{ji}B_{jj}B_{jk} \\ B_{ki}B_{kj}B_{kk} \end{bmatrix}, \quad [C]^{(e)} = \begin{Bmatrix} C_i \\ C_j \\ C_k \end{Bmatrix}^{(e)}, \tag{12}$$

where

$$B_{ii} = \frac{K}{4A}\left(b_i^2 + c_i^2\right) - \frac{PA}{6} + \frac{(\alpha L_{ij})_{\text{side}ij}}{3} + \frac{(\alpha L_{ki})_{\text{side}ki}}{3},$$

$$B_{ij} = \frac{K}{4A}(b_i b_j + c_i c_j) - \frac{PA}{12} + \frac{(\alpha L_{ij})_{\text{side}ij}}{6},$$

$$B_{ik} = \frac{K}{4A}(b_i b_k + c_i c_k) - \frac{PA}{12} + \frac{(\alpha L_{ki})_{\text{side}ki}}{3},$$

$$B_{jj} = \frac{K}{4A}\left(b_j^2 + c_j^2\right) - \frac{PA}{6} + \frac{(\alpha L_{jk})_{\text{side}jk}}{3} + \frac{(\alpha L_{ij})_{\text{side}ij}}{3},$$

$$B_{jk} = \frac{K}{4A}(b_j b_k + c_j c_k) - \frac{PA}{12} + \frac{(\alpha L_{jk})_{\text{side}jk}}{6},$$

$$B_{kk} = \frac{K}{4A}\left(b_k^2 + c_k^2\right) - \frac{PA}{6} + \frac{(\alpha L_{ki})_{\text{side}ki}}{3} + \frac{(\alpha L_{ij})_{\text{side}jk}}{3},$$

$$C_i = \frac{QA}{3} - \frac{\beta L_{ij}}{2} - \frac{\beta L_{ki}}{2}, \quad C_j = \frac{QA}{3} - \frac{\beta L_{jk}}{2} - \frac{\beta L_{ij}}{2},$$

$$C_k = \frac{QA}{3} - \frac{\beta L_{ki}}{2} - \frac{\beta L_{jk}}{2},$$

The element matrices were then assembled into the global matrices and vectors. The prescribed boundary conditions were implemented at the appropriate nodal points. The algebraic equations in the global assembled form were solved by the Gauss elimination procedure. These details may be found in the work of Allaire (1985).

Perturbation of the fluid mechanics problem

The finite element solution for the fluid mechanics problem may be reduced to the following equation in the unperturbed state:

$$[B][T] = [C] \tag{13}$$

The perturbed problem involving small fluctuations of the random variables may be written as

$$[\hat{B}][\hat{T}] = [\hat{C}] \tag{14}$$

where

$$\begin{aligned} [\hat{B}] &= [B] + \mathrm{d}[B] \\ [\hat{T}] &= [T] + \mathrm{d}[T] \\ [\hat{C}] &= [C] + \mathrm{d}[C] \end{aligned} \tag{15}$$

Therefore, we may write equation (10) as

$$\begin{aligned} [B]\mathrm{d}[T] &= [C] - [\hat{B}][T] - \mathrm{d}[B]\,\mathrm{d}[T] \\ &\cong \mathrm{d}x_i \frac{\partial [C]}{\partial x_i} - \mathrm{d}x_i \frac{\partial [B]}{\partial x_i}[T] \end{aligned} \tag{16}$$

In the last step in equation (16), we ignored the second order term $\mathrm{d}[B]\cdot\mathrm{d}[T]$. Here, x_i are the random variables. A simple form of the iterative algorithm is given by:

$$[B]\mathrm{d}[\hat{T}]^{n+1} = [\hat{C}] - [\hat{B}][\hat{T}]^n \tag{17}$$

$$[\hat{T}]^{n+1} = [\hat{T}]^n + \mathrm{d}[\hat{T}]^{n+1} \tag{18}$$

In order to start the iteration, we may use

$$[\hat{T}]^0 = [T]$$

The effect of variable properties may be included in equation (17). From equation (17), we may write:

$$[B]\mathrm{d}[\hat{T}]^n = [\hat{C}] - [\hat{B}][\hat{T}]^{n-1} \tag{19}$$

from equations (17) and (19) we may write

$$[B]\mathrm{d}[\hat{T}]^{n+1} = [B]\mathrm{d}[\hat{T}]^{n} - [\hat{B}]\mathrm{d}[\hat{T}]^{n} \tag{20}$$

From equation (20), we may write

$$d[\hat{T}]^{n+1} = [A]\mathrm{d}[\hat{T}]^{n} \tag{21}$$

where $[A] = [I] - [B]^{-1}\,[\hat{B}]$ is the amplification matrix. The iterative process will remain stable if the spectral radius of the amplification matrix $[A]$ is less than unity. This will be true when the imposed perturbations on the original element matrix are small.

Probability functions

Attention is now directed to the implementation of this probabilistic formulation in the design process. The necessary transition from the mathematical formulation above to a probabilistic model that yields the information relevant for multi-variate decision-making is described in this section. There are two alternatives for this task.

Joint probability model

The first joint probability density function (PDF) introduced here is an analytical probability model for criteria whose univariate distributions and their corresponding means and standard deviations are known. All necessary information for the model can be generated by the traditional probabilistic design process, using its output of univariate criterion distributions. A particular model for two criteria with normal distributions, has been introduced by Garvey and Tuab. Garvey further generated models for two criteria with combinations of normal and lognormal distributions, which are summarized in the work of Sundararajan (1995).

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}\exp\left\{\frac{1}{2\rho^2-2}\left[\left(\frac{x-\mu_x}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]\right\} \tag{22}$$

Note that the only information needed for the joint probability model consists of the means μ_X and μ_Y, the standard deviations σ_X and σ_Y, and the correlation coefficient ρ for the criteria X and Y. The model variables, x and y, are defined over the interval of all possible criterion values. The advantage of this model is the limited information needed, which makes it very flexible for use and application. For example, if only expert knowledge and no simulation/modeling is available in the early stages of design, educated guesses for the means, standard deviations, and the correlation coefficient can be used to execute the joint probability model. It also lends itself to use in combination with increasingly important fast probability integration (FPI) techniques.

Implementation of probabilistic procedure using FPI. FPI is a probabilistic analysis tool that implements a variety of methods for probabilistic analysis. The procedure follows the steps given below:

(1) identify the independent and uncorrelated design variables with uncertainties.

(2) quantify the uncertainties of these design variables with probability distributions based on expert opinion elicitation, historical data or benchmark testing.

(3) it is required that there is a response function that defines the relationship between the response and the independent variables.

(4) the FPI uses the responses generated to compute the cumulative distribution functions (CDF)/PDF and the corresponding sensitivities of the response.

Several methods are available in the FPI to compute a probabilistic distribution. In addition to obtaining the CDF/PDF of the response, the FPI provides additional information regarding the sensitivity of the response with respect to the primitive variables. They provide valuable information in controlling the scatter of the response variable. The random primitive variable with the highest sensitivity factor will yield the biggest payoff in controlling the scatter in that particular response variable. Such information is very useful to the test/design engineer in designing or interpreting the measured data.

Results and discussion

The history of the iterative algorithm is illustrated by means of two examples involving fluid flow through a circular pipe and slider bearing. We consider a 45° solution region for the circular pipe by considering of the symmetry. We assume that the fluid is water. The problem parameters are the following:

$$\frac{\partial p}{\partial z} = -34,474\,\mathrm{Pa}$$

$$\mu = 1,080 \times 10^{-6}\,\mathrm{N\,s/m^2}$$

$$R = 2.54 \times 10^{-5}\,\mathrm{m}$$

Figure 1 shows the solution region divided into nodes and elements. All random variables were assumed to be independent. A scatter of ± 10 percent was specified for all the primitive variables. Normal distribution was assumed for all random variable scatters.

The perturbed responses corresponding to the standard deviations from the mean for each of the random variables were computed. The CDF and the sensitivity factors of the volumetric flow rate were evaluated. CDF for the flow rate is shown in Figure 2. The sensitivity factors for the flow rate versus the

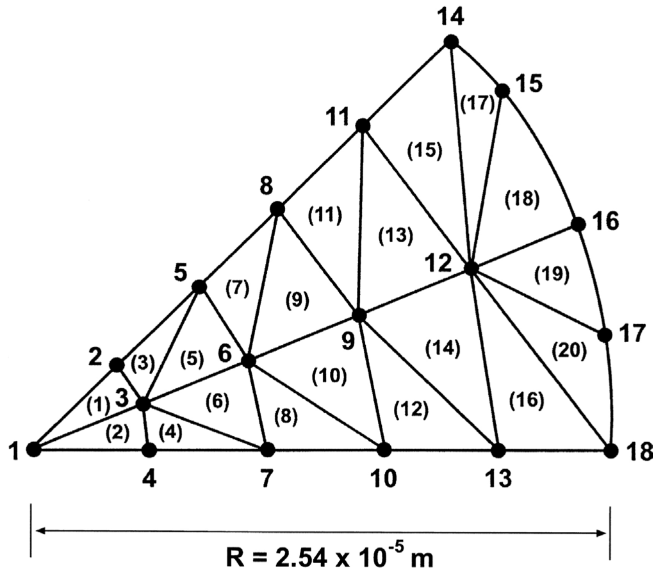

Figure 1.
Finite element model for pipe flow

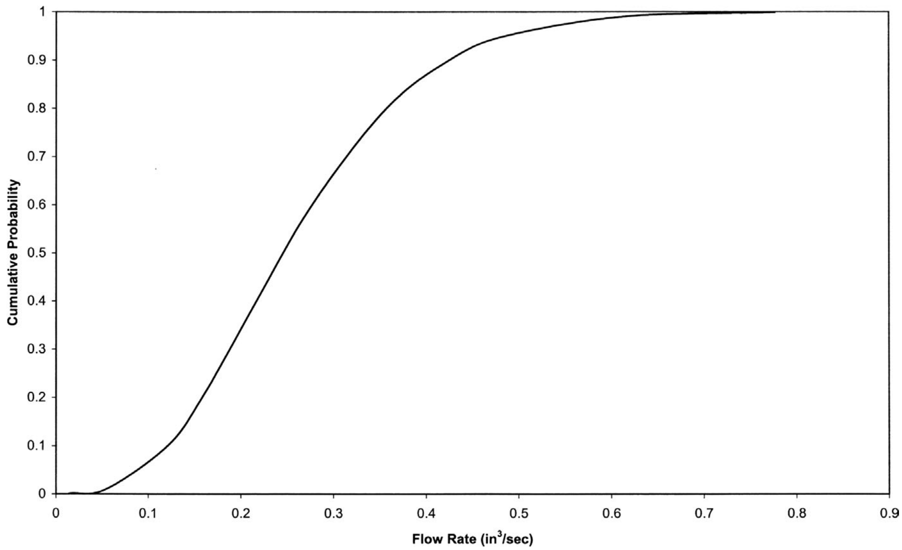

Figure 2.
Cumulative probability of the volumetric flow rate

primitive random variables are shown in Figure 3. We observe that the pipe radius has much of influence on volumetric flow rate. The sensitivities vary with the probability level because of the non-linearity of the problem. The viscosity and pressure gradients have much impact on flow rate at lower probability levels. Figure 4 shows the cumulative probability versus sensitivity

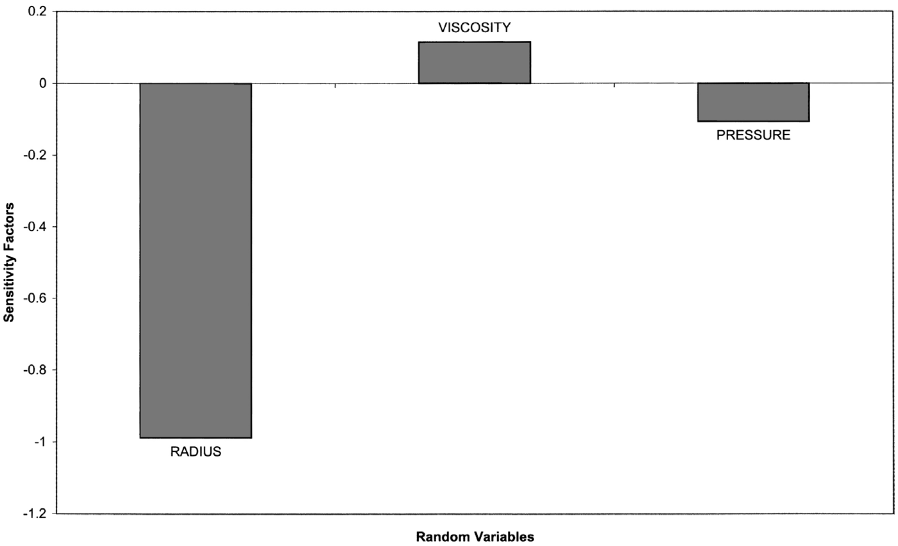

Figure 3.
Sensitivity factors versus random variables

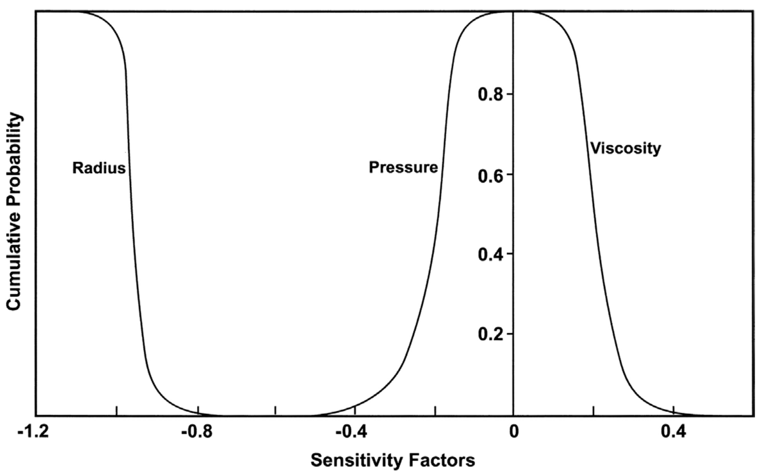

Figure 4.
Cumulative probability versus sensitivity factors

factors. It is clear from this plot that the pipe geometry has the most impact on flow rate.

Figure 5 shows the nodes and elements of the plane slider. The problem parameters are:

Length of the slider $(L) = 0.0762\,\text{m}$

Width of the slider $(W) = 0.0513\,\text{m}$

Slider velocity $(U) = 27.94\,\text{m/s}$

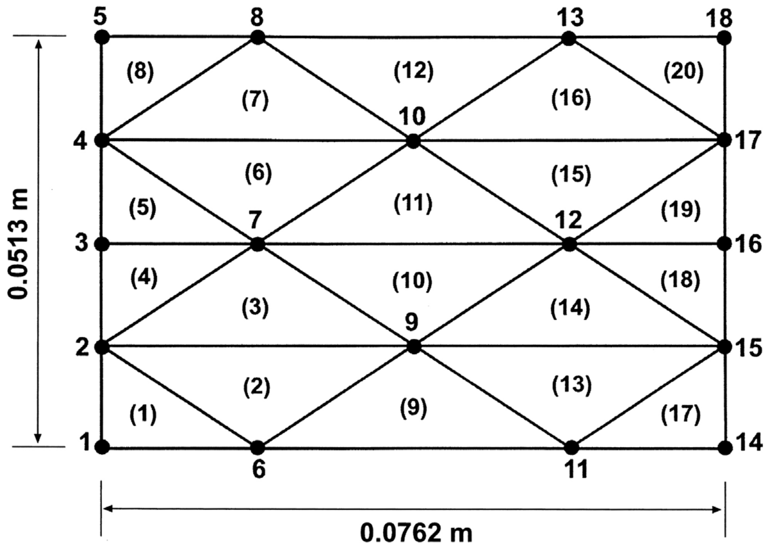

Figure 5.
Finite element model for slider bearing

Oil viscosity $(\mu) = 2{,}160 \times 10^{-5}\,\mathrm{N\,s/m^2}$

Oil film thickness $(h) = 0.0000762\,\mathrm{m}$

All random variables were assumed to be independent. A scatter of ± 10 percent was specified for all the variables. Normal distribution was assumed for all random variable scatters.

The perturbed responses corresponding to the standard deviations from the mean for each of the random variables were computed. The CDF and the sensitivity factors of the bearing load were evaluated. CDF for the bearing load is shown in Figure 6. The sensitivity factors for the load carrying capacity of the bearing versus the primitive random variables are shown in Figure 7. We observe that the oil film thickness has much influence on the force generated by the oil film at higher probability levels. The geometric variables of the slider bearing have equally important impact at lower probability levels. Figure 8 shows the cumulative probability versus sensitivity factors. It is clear from this plot that the film thickness has the most impact on the load carrying capacity of the bearing.

Conventional engineering design methods are deterministic. The components of a machine are considered as ideal systems and parameter optimization provide single point estimates of the system response. In reality, many engineering systems are stochastic where a probability assessment of the results is required. Probabilistic engineering design analysis assumes probability distributions of design parameters, instead of mean values only. This enables the designer to design for a specific reliability and hence maximize safety, quality and cost. The approaches for incorporating the

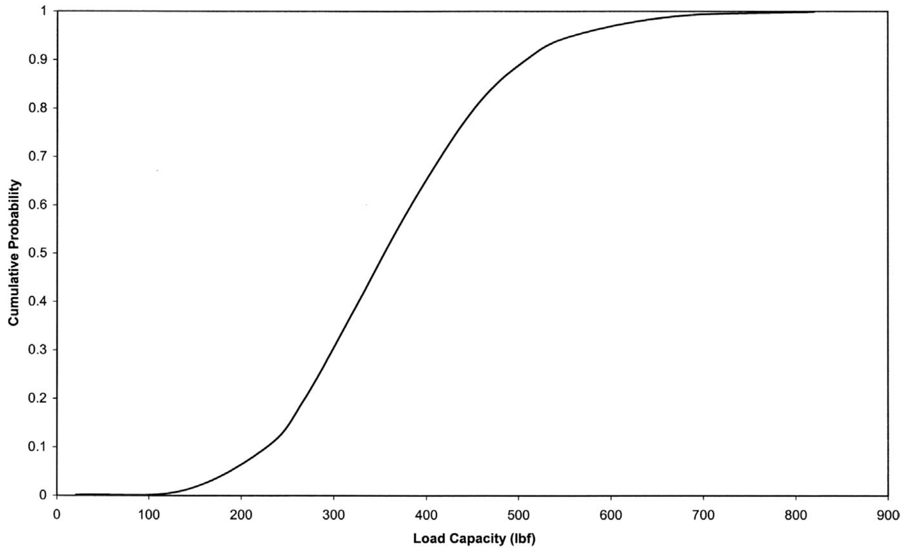

Figure 6.
Cumulative probability of bearing load capacity

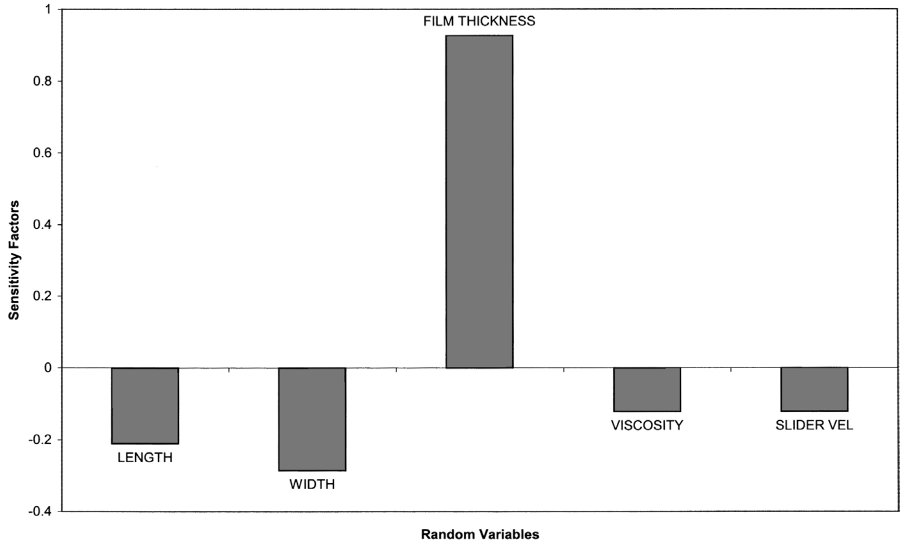

Figure 7.
Sensitivity factors versus random variables

probabilistic effects in design include the use of factors of safety, use of the worst-case design and use of probabilistic design. Utilizing the uncertainties in the estimations, deterministic engineering design uses factors of safety to assure that the nominal operational conditions does not come too close to the point where the system will fail. The approximation of minimum properties and maximum loads known as the absolute worst case gives information about

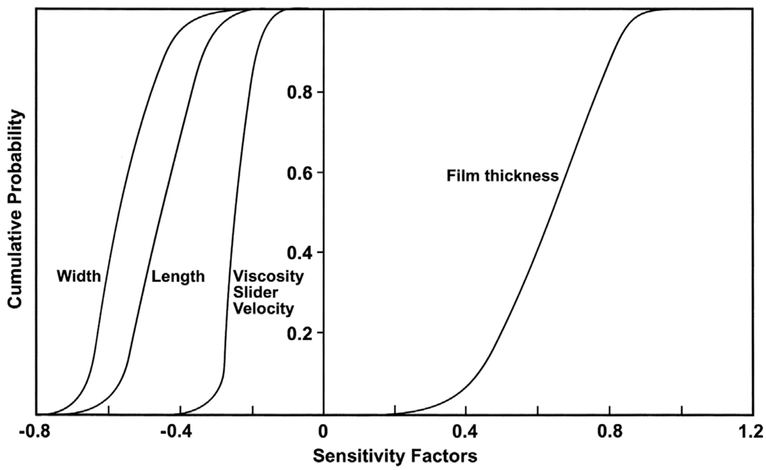

Figure 8.
Cumulative probability versus sensitivity factors

this critical point. This approach limits the optimization capability of a system and fails to provide important information about the system lifetime.

The design procedures of the advanced aerospace vehicles must account for uncertainties calculating the risk or reliability. These calculations will involve the probabilistic analysis. When compared with traditional factor of safety methods, probabilistic methods require additional inputs, but provide higher quality outputs. The uncertain or random variables are assumed to have a PDF. The output will be a PDF for the response quantities.

A robust design is one that has been created with a system of design tools that reduce product or process variability while guiding the performance toward an optimal setting. Robustness means achieving excellent performance under a wide range of operating conditions. All engineering systems function reasonably well under ideal conditions, but robust designs continue to function well when the conditions are non-ideal. Analytical robust design attempts to determine the values of design parameters which maximize the reliability of the product without tightening the material or environmental tolerances. Probabilistic design and robust design go hand in hand. In order to determine the domains of stability, the system has to be analyzed probabilistically.

Concluding remarks

In this paper, a non-deterministic method has been developed to support reliability-based design. The novelty in the paper is the probabilistic evaluation of the finite element solution for fluid flow. CDF and sensitivity factors were computed for volumetric flow rate in a pipe and load carrying capacity of a slider bearing due to the random variables. Evaluating the probability of risk and sensitivity factors will enable the identification of the most critical design variable in order to optimize the design and make it cost effective.

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